NON-PROPER ACTIONS OF THE FUNDAMENTAL GROUP OF A PUNCTURED TORUS

VIRGINIE CHARETTE

ABSTRACT. Given an affine isometry of $\mathbb{R}^3$ with hyperbolic linear part, its Margulis invariant measures signed Lorentzian displacement along an invariant spacelike line. In order for a group generated by hyperbolic isometries to act properly on $\mathbb{R}^3$, the sign of the Margulis invariant must be constant over the group. We show that, in the case when the linear part is the fundamental group of a punctured torus, positivity of the Margulis invariant over any finite generating set does not imply that the group acts properly. This contrasts with the case of a pair of pants, where it suffices to check the sign of the Margulis invariant for a certain triple of generators.

1. INTRODUCTION

A flat Lorentz manifold is a quotient $\mathbb{R}^3/\Gamma$, where Γ is a group of affine Lorentz transformations acting properly on $\mathbb{R}^3$. Lorentz transformations preserve a symmetric, non-degenerate bilinear form of signature $(2,1)$, denoted $\mathbb{B}(\cdot,\cdot)$.

Margulis [7, 8] proved that the fundamental group of a flat Lorentz manifold does not have to be virtually polycyclic, thus answering in the negative a question posed by Milnor. Milnor had asked whether Auslander's conjecture, that every crystallographic group must be amenable, can be generalized to non-cocompact groups.

Margulis' examples are affine groups whose linear parts are *Schottky groups*: free, discrete groups such that every non-identity element is hyperbolic. Drumm [3] (see also [2]) generalized his result by exhibiting fundamental domains for the actions of such groups; he also constructed examples containing parabolics.

While a group of isometries G acts properly on the hyperbolic plane if it is discrete, a group of affine transformations whose linear part is G might not act properly on $\mathbb{R}^3$, even though it acts freely. We are thus led to study *affine deformations* of a discrete group G, which are groups of affine transformations whose linear part is G. We seek conditions for an affine deformation of G to act freely and properly on $\mathbb{R}^n$, yielding a flat affine manifold whose fundamental group π is isomorphic to G.

We require that the projection of π onto its linear part be injective, which amounts to producing an affine deformation of G by assigning translational parts to a set of generators. Drumm-Goldman [5] determined geometric conditions for such an assignment to yield a proper deformation of G, when G is a Schottky group. It leads one to wonder, can properness of an affine deformation of a Schottky group be determined by conditions on some generating set of G?

An affine hyperbolic isometry admits a measure of signed Lorentzian displacement, called the *Margulis invariant*. For an affine group of hyperbolic isometries to act properly on $\mathbb{R}^3$, the Margulis invariant must be either always positive in the group, or always negative [7, 8].

It has been conjectured by Goldman-Margulis that this necessary condition is also sufficient. An interesting, further question is, if the Margulis invariant is positive for a finite set of elements in the group, can we deduce that the group must act properly on $\mathbb{R}^3$?

Jones [6] proved that when $G = \langle g, h \rangle$ is the fundamental group of a pair of pants, an affine deformation of the group $\Gamma = \langle \gamma, \eta \rangle$ acts properly and freely if and only if the Margulis invariants of γ, η and $\gamma\eta$ all have the same sign. This is achieved by showing that the translational parts assigned to g and h satisfy the Drumm-Goldman geometric condition.

In this paper, we show that Jones' result does not extend to the case when G is the fundamental group of a punctured torus. We elaborate on a construction by Drumm [4], which gives examples of groups $\langle \gamma, \eta \rangle$, where the Margulis invariant for both γ and η is positive, but is negative for $\gamma\eta$. Theorem 1.1, stated below, describes examples of affine groups $\langle \gamma, \eta \rangle$, with Schottky linear part, such that:

- the Margulis invariant of every word of length up to n is positive;
- the Margulis invariant of $\gamma\eta^n$ is negative.

We can construct such counterexamples for every n, with a fixed Schottky linear part.

Recall that every hyperbolic isometry w admits an attracting, future-pointing eigenvector (of Euclidean length one), which we denote by x_w^+. The fundamental group of a punctured torus has the following interesting property. If $\langle g, h \rangle$ is the fundamental group of a punctured torus, the set of vectors v such that $\mathbb{B}(\mathsf{v}, \mathsf{x}_{gh^i}^+) > 0$, for all $i \leq n$, intersects non-trivially with the set of vectors v such that $\mathbb{B}(\mathsf{v}, \mathsf{x}_{gh^{n+1}}^+) < 0$. This is not true if $\langle g, h \rangle$ is the fundamental group of a pair of pants.

The paper is organized as follows. Section 2 introduces notation. We define the Margulis invariant and describe some of its properties, including a useful addition formula.

Section 3 discusses relative positioning of eigenvectors for words in the Schottky group. We apply some basic ideas of symbolic dynamics on the boundary of the hyperbolic plane (see, for instance [1]). We show that there is a non-empty set of vectors v, such that:

- $\mathbb{B}(\mathsf{v}, \mathsf{x}^0_{gh^n}) < 0$;
- $\mathbb{B}(\mathsf{v}, \mathsf{x}^0_w) > 0$, for every word w of length up to n, except if the left factor of w is of the form ghg^{-1}.

This ensures that, choosing an appropriate factor of v to be the translational part of γ, the Margulis invariant is always positive, as long as there are no factors of the form ghg^{-1}.

In the second item, the case of words starting with ghg^{-1} poses a little bit of a difficulty, since the positioning of their eigenvectors potentially yield negative values of the Margulis invariant for words of small length. However, we may impose a certain technical condition on $\langle g, h \rangle$, called *Property C*; roughly speaking, Property C bounds the distance between the attracting eigenvector of gh^{-1} and the attracting eigenvector of ghg^{-1}. In the addition formula for the Margulis invariant, negative contributions from words of the form $ghg^{-1}w$ are then cancelled out by contributions from words of the form $gh^{-1}g^{-1}$. Property C thus ensures that the Margulis invariant is positive for *every* word up to length n.

Finally, in section 4, we prove the main theorem:

Theorem 1.1. *Let $G = \langle g, h \rangle$ be a transversal Schottky group having Property C. Then for every $n \geq 1$, G admits an affine deformation $< \gamma, \eta >$ that does not act properly on $\mathbb{A}^{2,1}$, such that the Margulis invariant of every word of length less than or equal to n is positive.*

2. Margulis invariant, properties

Let $\mathbb{R}^{2,1}$ denote the three-dimensional vector space with the indefinite, symmetric bilinear form defined as follows:

$$\mathbb{B}(\mathsf{x}, \mathsf{y}) = x_1 y_1 + x_2 y_2 - x_3 y_3,$$

where $\mathsf{x} = (x_1, x_2, x_3)$ and $\mathsf{y} = (y_1, y_2, y_3)$. A vector x is called *spacelike*, *timelike* or *lightlike*, respectively, if $\mathbb{B}(\mathsf{x}, \mathsf{x})$ is positive, negative or null.

The set of lightlike vectors is called the *lightcone*. The lightcone minus the zero vector admits two connected components, according to the sign of the third coordinate. A non-zero lightlike vector whose third coordinate is positive (resp. negative) is *future-pointing* (resp. *past-pointing*). Denote by $\mathfrak{F}$ the set of future-pointing lightlike rays.

The *Lorentz-perpendicular plane* of $\mathsf{v} \in \mathbb{R}^{2,1}$ is:

$$\mathsf{v}^\perp = \{\mathsf{x} \in \mathbb{R}^{2,1} \mid \mathbb{B}(\mathsf{x}, \mathsf{v}) = 0\}.$$

A vector v is *unit-spacelike* if $\mathbb{B}(\mathsf{v}, \mathsf{v}) = 1$. Denote by $\mathbb{S}^{2,1}$ the set of all unit-spacelike vectors:

$$\mathbb{S}^{2,1} = \{\mathsf{v} \in \mathbb{R}^{2,1} \mid \mathbb{B}(\mathsf{v}, \mathsf{v}) = 1\}.$$

If v is spacelike, its Lorentz-perpendicular plane intersects the lightcone in two lightlike lines.

Definition 2.1. Let $\mathsf{v} \in \mathbb{S}^{2,1}$. Choose x^+_v, $\mathsf{x}^-_\mathsf{v} \in \mathsf{v}^\perp$ as follows:

- x_v^+, $x_v^- \in \mathfrak{F}$;
- x_v^+, x_v^- are of Euclidean length one;
- (v, x_v^-, x_v^+) is a positively oriented basis.

Then (v, x_v^-, x_v^+) is called the *null frame* relative to v.

(Requiring that $x_v^\pm$ are unit length simply ensures uniqueness of choice.)

The *Lorentz cross-product* is the unique bilinear transformation $\boxtimes : \mathbb{R}^{2,1} \times \mathbb{R}^{2,1} \longrightarrow \mathbb{R}^{2,1}$ such that:

$$\mathbb{B}(u, v \boxtimes w) = \det[u \ v \ w].$$

In particular, $\mathbb{B}(v, v \boxtimes w) = 0$. It is easily seen that $v \in \mathbb{S}^{2,1}$ is a positive scalar multiple of $x_v^- \boxtimes x_v^+$.

We may identify $\mathbb{R}^{2,1}$ with $\mathbb{A}^{2,1}$, the three-dimensional affine space whose space of translations is $\mathbb{R}^{2,1}$.

2.1. Isometries.
Let $\mathrm{Aff}(\mathbb{A}^{2,1})$ be the group of affine isometries of $\mathbb{A}^{2,1}$. Via the identification between $\mathbb{R}^{2,1}$ and $\mathbb{A}^{2,1}$, any $\gamma \in \mathrm{Aff}(\mathbb{A}^{2,1})$ may be written as follows:

$$\gamma(x) = g(x) + v_g,$$

where $g \in \mathrm{O}(2,1)$ and $v_g \in \mathbb{R}^{2,1}$; g is called the *linear part* of γ and v_g, its *translational part*.

We shall call γ an *affine deformation of g*. More generally, $\Gamma \subset \mathrm{Aff}(\mathbb{A}^{2,1})$ is called an *affine deformation* of G if its linear part is G. An affine deformation is obtained from a linear group by assigning translational parts to its generators.

Isometries in $\mathrm{O}(2,1)$ either preserve each connected component of the lightcone or interchange them. Let $\mathrm{SO}(2,1)^0$ denote the connected component of the identity in $\mathrm{O}(2,1)$; $\mathrm{SO}(2,1)^0$ consists of orientation- and time orientation-preserving isometries.

Let $\gamma \in \mathrm{Aff}(\mathbb{A}^{2,1})$; suppose its linear part g lies in $\mathrm{SO}(2,1)^0$. Both γ and g are called *hyperbolic* if g admits three distinct eigenvalues. Denote its smallest eigenvalue by λ_g; it must be positive and less than 1 and the eigenvalues of g are thus $\lambda_g, 1, \lambda_g^{-1}$. The $\lambda_g^{\pm 1}$ eigenvectors must be lightlike and the 1-eigenvectors, spacelike. Choose x_g^+, x_g^- and x_g^0 as follows:

- x_g^- is a λ_g-eigenvector and x_g^+, a λ_g^{-1}-eigenvector;
- $x_g^0 \in \mathbb{S}^{2,1}$ is an eigenvector such that $x_{x_g^0}^\pm = x_g^\pm$.

This is the null frame of x_g^0 and is uniquely determined by g. Note that $x_{g^{-1}}^0 = -x_g^0$ and that for any isometry $h \in \mathrm{SO}(2,1)^0$:

$$x_{hgh^{-1}}^0 = h x_g^0.$$

2.2. The Margulis invariant.
Suppose $\gamma \in \mathrm{Aff}(\mathbb{A}^{2,1})$ is hyperbolic, with linear part g. Then γ preserves a unique line in $\mathbb{A}^{2,1}$ that is parallel to $x_{(}^0 g)$. (It is the only γ-invariant line if γ acts freely.) Since (x_g^-, x_g^+, x_g^0) forms a basis for $\mathbb{R}^{2,1}$, the following definition is independent of the choice of p.

Definition 2.2. Let $\gamma \in \mathrm{Aff}(\mathbb{A}^{2,1})$ be hyperbolic with linear part g. The *Margulis invariant* of γ is:

$$\text{(1)} \qquad \alpha(\gamma) = \mathbb{B}(\gamma(p) - p, \mathsf{x}_g^0),$$

where p is an arbitrary point in $\mathbb{A}^{2,1}$.

In particular, if v_g is the translational part of γ, then $\alpha(\gamma) = \mathbb{B}(\mathsf{v}_g, \mathsf{x}_g^0)$. This invariant has the following properties:

- $\alpha(\gamma^{-1}) = \alpha(\gamma)$;
- For any $\eta \in \mathrm{Aff}(\mathbb{A}^{2,1})$, $\alpha(\eta\gamma\eta^{-1}) = \alpha(\gamma)$.

The Margulis invariant yields a useful criterion for determining that an action is non-proper.

Lemma 2.3. *(Margulis [7, 8]) Let γ, $\eta \in \mathrm{Aff}(\mathbb{A}^{2,1})$ be hyperbolic transformations such that $\alpha(\gamma)\alpha(\eta) < 0$. Then $\langle \gamma, \eta \rangle$ does not act properly on $\mathbb{A}^{2,1}$.*

2.3. **The Margulis invariant of a cyclically reduced word.** Suppose $g = \langle g_1, \ldots, g_n \rangle$ is freely generated by the g_i's. Then every $g \in G$ can be uniquely written as a *word* in the generators:

$$w = g_{i_k}^{j_k} g_{i_{k-1}}^{j_{k-1}} \ldots g_{i_1}^{j_1},$$

where $1 \leq i_l \leq n$, $j_l = \pm 1$ and $g_{i_l} \neq g_{i_{l+1}}^{-1}$. This last condition states that w is a *reduced word*. The length of a reduced word is well-defined. We call g_{i_k} the *terminal letter* of w and g_{i_1}, its *initial letter*. (This is consistent with a left action of $\mathrm{Aff}(\mathbb{A}^{2,1})$ on $\mathbb{A}^n$.)

We will say that a reduced word g is *cyclically reduced* if g^2 is reduced. In other words, g is not the conjugate of a reduced word of shorter length.

Lemma 2.4. *(Drumm-Goldman [5]) Let $\Gamma = \langle \gamma_1, \ldots, \gamma_n \rangle$ be an affine deformation of a group that is freely generated by hyperbolic elements. Denote the linear part of γ_i by g_i and its translational part by v_{g_i}. Suppose $\gamma \in \Gamma$ is a cyclically reduced hyperbolic isometry and write its linear part as the reduced word:*

$$w = g_{i_1}^{j_1} g_{i_2}^{j_2} \ldots g_{i_m}^{j_m},$$

where $j_i = \pm 1$. Then:

$$\text{(2)} \qquad \alpha(\gamma) = \sum_{k=1}^{n} \mathbb{B}(\mathsf{v}_{g_{i_k}}, \mathsf{x}_{w_k}^0),$$

where w_k is a cyclically reduced word whose terminal letter is g_{i_k}.

Note that we have written the terminal letter as $g_{i_1}^{j_1}$, in order to simplify notation in the proof.

Proof. To simplify notation, set:

$$h_k = g_{i_k}^{j_k}$$
$$\mathsf{v}_k = \mathsf{v}_{h_k},$$

for $1 \leq k \leq m$. Thus:

$$\alpha(\gamma) = \mathbb{B}(\mathsf{v}_1, \mathsf{x}_w^0) + \sum_{k=2}^{n} \mathbb{B}(h_1 \ldots h_{k-1}(\mathsf{v}_k), \mathsf{x}_w^0)$$

$$= \mathbb{B}(\mathsf{v}_1, \mathsf{x}_w^0) + \sum_{k=2}^{n} \mathbb{B}(\mathsf{v}_k, (h_1 \ldots h_{k-1})^{-1} \mathsf{x}_w^0)$$

$$= \mathbb{B}(\mathsf{v}_1, \mathsf{x}_w^0) + \sum_{k=2}^{n} \mathbb{B}(\mathsf{v}_k, \mathsf{x}_{h_k \ldots h_m h_1 \ldots h_{k-1}}^0),$$

since $w_2^{-1} \mathsf{x}_{w_1}^0 = \mathsf{x}_{w_2^{-1} w_1 w_2}^0$.

Now consider each summand $\mathbb{B}(\mathsf{v}_k, \mathsf{x}_{h_k \ldots h_m h_1 \ldots h_{k-1}}^0)$. If $j_k = 1$, then $h_k = g_{i_k}$ and $\mathsf{v}_k = \mathsf{v}_{g_{i_k}}$ so that the summand is of the desired form. Moreover, w_k is reduced because γ is cyclically reduced, and cyclically reduced because γ is reduced.

If $j_k = -1$, then the corresponding summand is:

$$\mathbb{B}(-g_{i_k}^{-1}(\mathsf{v}_{g_{i_k}}), \mathsf{x}_{g_{i_k}^{-1} \ldots h_m h_1 \ldots h_{k-1}}^0) = -\mathbb{B}(\mathsf{v}_{g_{i_k}}, \mathsf{x}_{w_k' g_{i_k}^{-1}}^0)$$

$$= \mathbb{B}(\mathsf{v}_{g_k}, \mathsf{x}_{g_{i_k}(w_k')^{-1}}^0),$$

where $w_k' = h_{k+1} \ldots h_m h_1 \ldots h_{k-1}$. As above, $w_k = g_{i_k}(w_k')^{-1}$ is both reduced and cyclically reduced. $\square$

Example 2.5. Let $\gamma, \eta \in \mathrm{Aff}(\mathbb{A}^{2,1})$ be two hyperbolic elements with respective linear parts g, h and respective translational parts v_g, v_h. Then:

$$\alpha(\gamma\eta) = \mathbb{B}(\mathsf{v}_g, \mathsf{x}_{gh}^0) + \mathbb{B}(\mathsf{v}_h, \mathsf{x}_{hg}^0)$$
$$\alpha(\gamma\eta^{-1}) = \mathbb{B}(\mathsf{v}_g, \mathsf{x}_{gh^{-1}}^0) + \mathbb{B}(\mathsf{v}_h, \mathsf{x}_{hg^{-1}}^0).$$

More generally, for $n \geq 1$:

$$(3) \qquad \alpha(\gamma\eta^n) = \mathbb{B}(\mathsf{v}_g, \mathsf{x}_{gh^n}^0) + \sum_{k=0}^{n-1} \mathbb{B}(\mathsf{v}_h, \mathsf{x}_{h^{n-k}gh^k}^0).$$

Following Drumm [4], we can use the formula for $\alpha(\gamma\eta)$ to construct non-proper actions. Indeed, the intersection of the half-spaces:

$$\{\mathsf{v} \in \mathbb{R}^{2,1} \mid \mathbb{B}(\mathsf{v}, \mathsf{x}_g^0) > 0\} \bigcap \{\mathsf{v} \in \mathbb{R}^{2,1} \mid \mathbb{B}(\mathsf{v}, \mathsf{x}_{gh}^0) < 0\}$$

is convex and non-empty. Choose any v in this intersection and choose the translational part of η to be any v_h such that $\alpha(\eta)$ is positive. Then set $\mathsf{v}_g = k\mathsf{v}$, where k is large enough so that $\alpha(\gamma\eta) < 0$. Observe that $\alpha(\gamma)$ and $\alpha(\eta)$ are both positive, but by Lemma 2.3, the group generated by γ and η does not act properly, since the subgroup $\langle \gamma, \gamma\eta \rangle$ does not act properly.

3. Symbolic dynamics on the boundary at infinity

We now discuss the relative positioning of attracting null eigenvectors of words, based on the positioning of the eigenvectors of a generating set for the group. In the case of a *Schottky group*, defined below, we can use Brouwer's fixed point theorem.

Denote the *sign* of an integer i by $\sigma(i)$ and the closure of a set A by $cl(A)$. In what follows, the closure of subsets of $\mathfrak{F}$, the set of future-pointing lightlike rays, will always be taken relative to $\mathfrak{F}$.

Definition 3.1. A connected set of future-pointing lightlike rays is called a *conical interval*. A *conical neighborhood* of $\mathsf{x} \in \mathfrak{F}$ is a conical interval containing x.

Definition 3.2. Let $g_1, \ldots, g_n \in \mathrm{SO}(2,1)^0$. Then $\langle g_1, \ldots, g_n \rangle$ is a *Schottky group* if there exist $2n$ disjoint closed conical intervals $A_i^{\pm}$ such that, for $1 \le i \le n$:
$$g_i(A_i^-) = cl(\mathfrak{F} - A_i^+).$$
The generators $g_1, \ldots, g_n$ are called *Schottky generators* of G and the conical intervals $A_i^{\pm}$ are called a *Schottky system*.

See Figure 1.

Note that $\mathsf{x}_{g_i}^{\pm} \in A_i^{\pm}$. In fact, as is illustrated in Figure 1, Brouwer's fixed point theorem implies that when $w = g_{i_n}^{j_n} \ldots g_{i_1}^{j_1}$ is a cyclically reduced word, then $\mathsf{x}_w^+ \in A_{i_n}^{\sigma(j_n)}$ and $\mathsf{x}_w^- \in A_{i_1}^{\sigma(-j_1)}$.

Equivalently, a finitely generated subgroup of $O(2,1)$ is Schottky if and only if it is free, discrete and purely hyperbolic.

Let $U \ne \emptyset$, $\mathfrak{F}$ be a closed conical interval. Denote by x_U^-, x_U^+, respectively, the (Euclidean) unit-length lightlike vectors spanning the boundary of U, such that $\mathbb{B}(\mathsf{x}, \mathsf{x}_U^- \boxtimes \mathsf{x}_U^+) > 0$ for $\mathsf{x} \in U$.

If U^-, U^+ is a disjoint pair of closed and non-trivial conical intervals, define:
$$\mathcal{C}[U^-, U^+] = \{\mathsf{v} \in \mathbb{R}^{2,1} \mid \mathbb{B}(\mathsf{v}, \mathsf{u}) > 0 \text{ for all } \mathsf{u} \in \mathbb{S}^{2,1} \text{ such that}$$
$$\mathsf{x}_\mathsf{u}^- \in U^-, \ \mathsf{x}_\mathsf{u}^+ \in U^+\}.$$

This is a non-empty intersection of half-spaces. In fact, it is simply the intersection of four half-spaces, namely, those determined by $\mathsf{x}_{U^-}^{\pm} \boxtimes \mathsf{x}_{U^+}^{\pm}$.

Lemma 3.3. *Let* U^-, U^+ *be closed, disjoint, non-trivial conical intervals. Then* $\mathcal{C}[U^-, U^+]$ *is the cone of positive linear combinations of* $\mathsf{x}_{U^-}^+$, $\mathsf{x}_{U^+}^-$, $-\mathsf{x}_{U^+}^+$ *and* $-\mathsf{x}_{U^-}^-$.

$\square$

See Figure 2. In particular, if W is a set of words g such that $\mathsf{x}_g^- \in U^-$ and $\mathsf{x}_g^+ \in U^+$, then every $\mathsf{v} \in \mathcal{C}[U^-, U^+]$ satisfies $\mathbb{B}(\mathsf{v}, \mathsf{x}_g^0) > 0$, for every $g \in W$.
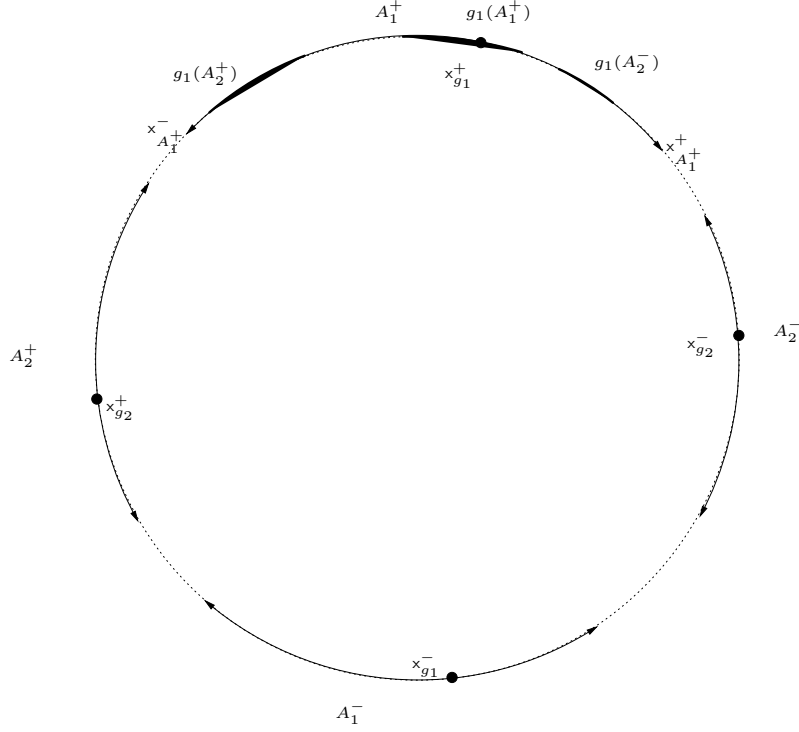
FIGURE 1. A Schottky system for $\langle g_1, g_2 \rangle$. Since $g_1(B) \subset A_1^+$ for $B = A_1^+$, $A_2^\pm$, $\mathsf{x}_h^+ \in A_1^+$ for $h = g_1^2$, $g_1 g_2$, $g_1 g_2^{-1}$.

Example 3.4. Let $G = \langle g, h \rangle$ be a Schottky group with Schottky system $A_g^\pm$, $A_h^\pm$. Set $U^+ = A_h^+$ and let U^- be the smallest (closed) conical interval containing $A_h^- \bigcup A_g^\pm$. If $\mathsf{v} \in \mathcal{C}[U^-, U^+]$:

$$\mathbb{B}(\mathsf{v}, \mathsf{x}_w^0) > 0,$$

for every word w with terminal letter h. Drumm-Goldman [5] construct affine deformations of Schottky groups that act properly on $\mathbb{A}^{2,1}$, using this idea.

3.1. **Ordering on the future lightcone.** Given a Schottky subgroup of $SO(2,1)^0$, we seek an ordering of the attracting eigenvectors in the Schottky intervals. Let $\mathsf{x}, \mathsf{y}, \mathsf{z} \in \mathfrak{F}$; we write:

$$\mathfrak{O}[\mathsf{x}, \mathsf{y}, \mathsf{z}]$$

if and only if $(\mathsf{x}, \mathsf{y}, \mathsf{z})$ is a right-handed positively-oriented basis, i.e. $\det([\mathsf{x}\,\mathsf{y}\,\mathsf{z}]) > 0$. More generally, $\mathfrak{O}[\mathsf{v}_1, \dots, \mathsf{v}_k]$ if and only if $\mathfrak{O}[\mathsf{v}_{i_1}, \mathsf{v}_{i_2}, \mathsf{v}_{i_3}]$ for every triple $1 \leq i_1 < i_2 < i_3 \leq k$.

We extend the notation to conical intervals in the obvious manner. Thus, if U is a conical interval, we will write $\mathfrak{O}[\mathsf{x}, U, \mathsf{z}]$ to mean that $\mathfrak{O}[\mathsf{x}, \mathsf{y}, \mathsf{z}]$ for every $\mathsf{y} \in U$.

Obviously, $\mathfrak{O}[\mathsf{x}, \mathsf{y}, \mathsf{z}]$ if and only if $\mathfrak{O}[\mathsf{y}, \mathsf{z}, \mathsf{x}]$.
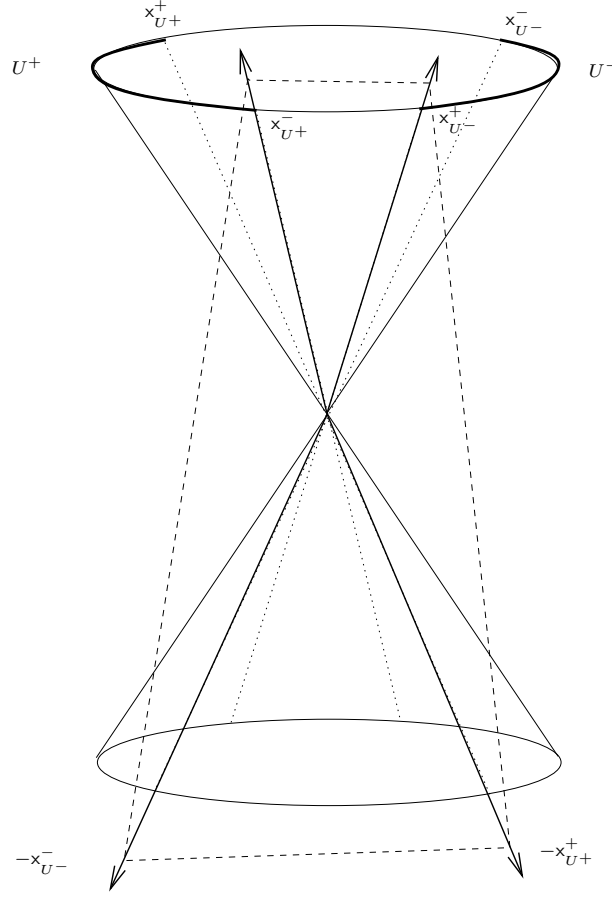
FIGURE 2. The cone $\mathcal{C}[U^-, U^+]$, which is the set of positive linear combinations of $x^-_{U^+}$, $x^+_{U^-}$, $-x^-_{U^-}$ and $-x^+_{U^+}$. For any u such that $x^-_u \in U^-$ and $x^+_u \in U^+$, and any $v \in \mathcal{C}[U^-, U^+]$, $\mathbb{B}(u, v) > 0$.

Lemma 3.5. *Let* $x, y, z \in \mathfrak{F}$. *Then* $\mathfrak{O}[x, y, z]$ *if and only if* $\mathbb{B}(x, y \boxtimes z) > 0$.

$\square$

If $g \in \mathrm{SO}(2, 1)^0$, then g is orientation-preserving; thus $\mathfrak{O}[x, y, z]$ if and only if $\mathfrak{O}[g(x), g(y), g(z)]$.

We will order vectors in a conical interval according to one of its boundary components. Note that if A is a conical interval and x, $y \in A$,

$$\mathfrak{O}[x^+_A, x, y] \iff \mathfrak{O}[x^-_A, x, y],$$

which both mean that, starting at x^+_A and moving counterclockwise, we meet x, y and then x^-_A.

Recall that if $G = \langle g_1, \ldots, g_n \rangle$ is a Schottky group, $x^+_w \in A^\pm_i$ for every reduced word w with terminal letter $g^{\pm 1}_i$. The following lemma yields an

ordering of the attracting eigenvectors of cyclically reduced words of the form $h^{\pm 1} g_{i_1}^{\pm 1} w_1$ and $h g_{i_2}^{\pm 1} w_2$, where $1 \le i_1, i_2 \le n$ and j_1, j_2 are non-zero integers. For simplicity, we write $g_1 = g_{i_1}^{\pm 1}$ and $g_2 = g_{i_2}^{\pm 1}$. This is allowed, since any Schottky generator can be replaced by its inverse and the generators can be reindexed.

Lemma 3.6. *Let $G = \langle g_1, \ldots, g_n \rangle$ be a Schottky group and let $h = g_i^k$, for some $1 < i < n$ and $k = \pm 1$. Set $A = A_i^{\sigma(k)}$. Let w_1, w_2 be non-trivial words in G such that $h g_1 w_1$ and $h g_2 w_2$ are cyclically reduced. Then:*

$$\mathfrak{O}[\mathsf{x}_A^+, \mathsf{x}_{h g_1 w_1}^+, \mathsf{x}_{h g_2 w_2}^+] \Longleftrightarrow \mathfrak{O}[\mathsf{x}_h^-, \mathsf{x}_{g_1}^+, \mathsf{x}_{g_2}^+].$$

Proof. Set $B_j = g_j(w_j(A))$, $j = 1, 2$ and $A^- = A_i^{\sigma(-k)}$. Then $B_j \subset A_j^+$, which is disjoint from A^-. Thus $\mathfrak{O}[\mathsf{x}_h^-, B_1, B_2]$ makes sense and:

$$\mathfrak{O}[\mathsf{x}_h^-, \mathsf{x}_{g_1}^+, \mathsf{x}_{g_2}^+] \Longleftrightarrow \mathfrak{O}[\mathsf{x}_h^-, B_1, B_2]$$
$$\Longleftrightarrow \mathfrak{O}[\mathsf{x}_{A^-}^-, B_1, B_2]$$
$$\Longleftrightarrow \mathfrak{O}[\mathsf{x}_A^+, h(B_1), h(B_2)],$$

since $h(\mathsf{x}_{A^-}^-)$ is parallel to x_A^+. By Brouwer's fixed point theorem, $h(B_i)$ contains the attracting eigenvector of $h g_i w_i$ and the result follows. $\square$

Corollary 3.7. *Using the hypotheses of Lemma 3.6, let $w \in G$ be a word with terminal letter $g = g_i^k$, such that $w_1' = w h g_1 w_1$ and $w_2' = w h g_2 w_2$ are cyclically reduced. Then:*

$$\mathfrak{O}[\mathsf{x}_{A'}^+, \mathsf{x}_{w_1'}^+, \mathsf{x}_{w_2'}^+] \Longleftrightarrow \mathfrak{O}[\mathsf{x}_h^-, \mathsf{x}_{g_1}^+, \mathsf{x}_{g_2}^+],$$

where $A' = A_i^{\sigma(k)}$.

Proof. This follows from Lemma 3.6 and the observation that $\mathfrak{O}[\mathsf{x}_{A''}^-, \mathsf{x}_{h g_1 w_1}^+, \mathsf{x}_{h g_2 w_2}^+]$ if and only if $\mathfrak{O}[\mathsf{x}_A^+, \mathsf{x}_{h g_1 w_1}^+, \mathsf{x}_{h g_2 w_2}^+]$, where $A'' = A_i^{\sigma(-k)}$. $\square$

3.2. Transversal isometries.

Definition 3.8. Suppose $< g, h > \subset \mathrm{SO}(2,1)^0$ is a Schottky group. The group is said to be *transversal* if the planes $\mathsf{x}_g^{0\perp}$ and $\mathsf{x}_h^{0\perp}$ intersect in a timelike line.

Thinking of the interior of the projectivized lightcone as the hyperbolic plane, this means that $< g, h >$ is the fundamental group of a punctured torus.

For the remainder of this section, we will assume $\langle g, h \rangle$ to be transversal. Taking h^{-1} if necessary, we may further assume that:

$$\mathfrak{O}[\mathsf{x}_g^-, \mathsf{x}_h^-, \mathsf{x}_g^+].$$

Moving counterclockwise on the future lightcone and starting at x_g^-, we meet, in order: x_h^-, x_g^+ and finally, x_h^+.
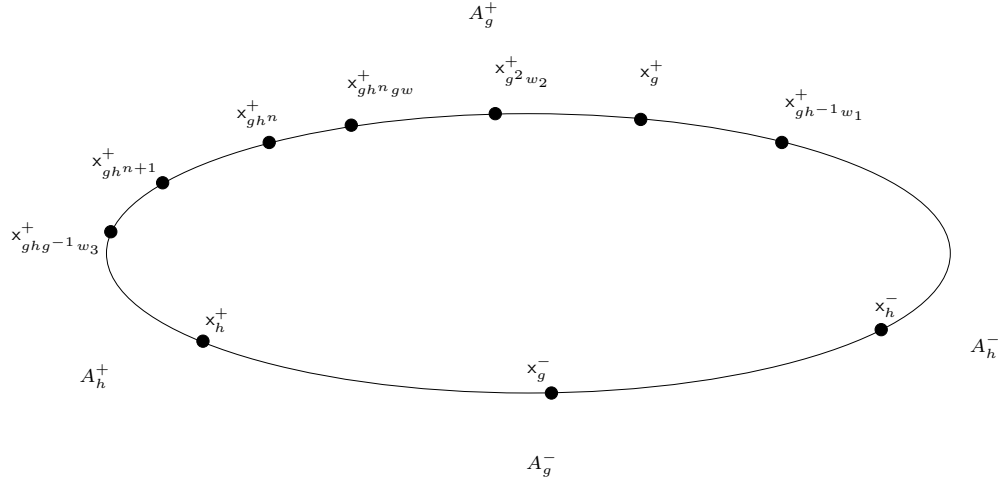
FIGURE 3. The ordering of some attracting eigenvectors in the case of transversal Schottky generators g and h.

Lemma 3.9. *Let $\langle g, h \rangle$ be a transversal Schottky group with Schottky system $\{A_g^{\pm}, A_h^{\pm}\}$, with $\mathfrak{O}[\mathsf{x}_g^-, \mathsf{x}_h^-, \mathsf{x}_g^+]$. Set $\mathsf{x}^+ = \mathsf{x}_{A_g^+}^+$. Then the following hold:*

(1) $\mathfrak{O}[\mathsf{x}^+, \mathsf{x}_{gh^{-1}w_1}^+, \mathsf{x}_{g^2 w_2}^+, \mathsf{x}_{ghw_3}^+]$ for every w_1, w_2, w_3 yielding cyclically reduced words;

(2) $\mathfrak{O}[\mathsf{x}^+, \mathsf{x}_{gh^i gw}^+, \mathsf{x}_{gh^{n+1}}^+]$, $i \leq n$, for any word w such that $gh^i gw$ is cyclically reduced;

(3) $\mathfrak{O}[\mathsf{x}^+, \mathsf{x}_{gh^i}^+, \mathsf{x}_{gh^{i+j}}^+]$, for all $i \geq 0$ and $j > 0$.

See Figure 3.

Proof. Items 1 and 2 follow from Lemma 3.6.

To prove Item 3, consider the conical interval U contained in A_g^+ bounded by $\mathsf{x}_{gh^i}^+$ and $\mathsf{x}_{A_g^+}^-$. Then $h^j(U) \subset A_h^+$ and thus, $\mathfrak{O}[\mathsf{x}_{A_g^+}^+, h^j(U), \mathsf{x}_{gh^i}^+]$. It follows that $\mathfrak{O}[\mathsf{x}^+, gh^{i+j}(U), \mathsf{x}_{gh^i}^+]$. By Brouwer's fixed point theorem, $\mathsf{x}_{gh^{i+j}}^+ \in gh^{i+j}(U)$. $\square$

Let U^- be the smallest conical interval containing $A_h^{\pm}$ and A_g^-. Next, for each $n \geq 1$, set U_n^+ to be the conical interval contained in A_g^+ bounded by $\mathsf{x}_{A_g^+}^+$ and $\mathsf{x}_{gh^{n-1}}^+$. Thus:

$$\mathsf{x}_{U^-}^- = \mathsf{x}_{A_h^-}^- \qquad\qquad \mathsf{x}_{U^-}^+ = \mathsf{x}_{A_h^+}^+$$
$$\mathsf{x}_{U_n^+}^- = \mathsf{x}_{gh^{n-1}}^+ \qquad\qquad \mathsf{x}_{U_n^+}^+ = \mathsf{x}_{A_g^+}^+.$$

By Lemma 3.9, if $\mathsf{v} \in \mathcal{C}[U^-, U_n^+]$, then $\mathbb{B}(\mathsf{v}, \mathsf{x}_g^0) > 0$ and, furthermore, $\mathbb{B}(\mathsf{v}, \mathsf{x}_w^0) > 0$ for every cyclically reduced word w of length less than or equal to $n + 1$, with terminal factor g^k, $k \geq 2$, $gh^i g$ or gh^{-1}.
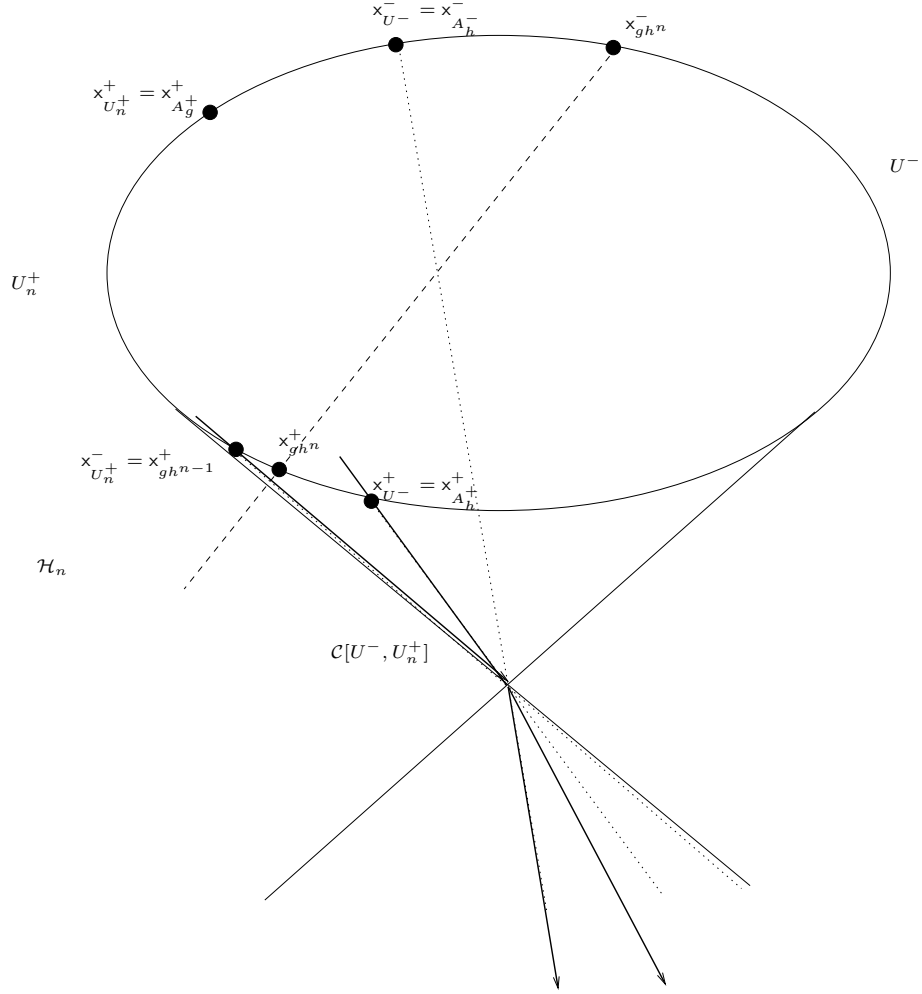
FIGURE 4. The cone $\mathcal{C}[U^-, U_n^+]$, bounded by the darker lines, intersects non-trivially with the half-space $\mathcal{H}_n$, bounded by the dashed line.

Now, for each $n \geq 1$, set:

$$\mathcal{H}_n = \{\mathsf{v} \in \mathbb{R}^{2,1} \mid \mathbb{B}(\mathsf{v}, \mathsf{x}_{gh^n}^0) < 0\}.$$

Observe that $\mathcal{H}_n \bigcap \mathcal{C}[U^-, U_n^+]$ is non-empty, since both sets contain the conical interval in A_g^+ bounded by $\mathsf{x}_{gh^{n-1}}^+$ and $\mathsf{x}_{gh^n}^+$. (See Figure 4.)

3.3. **Words of the form** $gh^i g^{-1} w$. Our goal is to assign translational parts v_g, v_h to g, h respectively, to obtain an affine deformation of a Schottky group $\Gamma = \langle \gamma, \eta \rangle$, such that

- $\alpha(\omega) > 0$ for every $\omega \in \Gamma$ of length less than or equal to n, and
- $\alpha(\gamma \eta^n) < 0$.

By Lemma 2.4:

$$\alpha(\omega) = \sum_{k=1}^{n} \mathbb{B}(\mathsf{v}_k, \mathsf{x}_{w_k}^0), \tag{4}$$

where $\mathsf{v}_k = \mathsf{v}_g$ (resp. $\mathsf{v}_k = \mathsf{v}_h$) and w_k is a cyclically reduced word whose terminal letter is g (resp. h).

We can start by choosing v_h such that $\mathbb{B}(\mathsf{v}_h, \mathsf{x}_w^0) > 0$ for every cyclically reduced word w with terminal letter h. Then, we can choose any $\mathsf{v} \in \mathcal{H}_n \bigcap \mathcal{C}[U^-, U_n^+]$, then set v_g to be a scalar multiple of v satisfying $\alpha(\gamma\eta^n) < 0$.

If every summand in Equation (4) were positive when the length of γ is at most n, then we would be done. And $\mathbb{B}(\mathsf{v}_g, \mathsf{x}_w^0)$ is indeed positive for many choices of w. But our choice of v_g yields a negative value for $\mathbb{B}(\mathsf{v}_g, \mathsf{x}_{gh^ig^{-1}}^0)$. As a matter of fact, for n large enough, it is impossible to force all the summands in Equation (4) to be positive.

However, it is possible that every negative summand is cancelled out by a positive summand, in light of the following observation.

Lemma 3.10. *Write $\alpha(\omega)$ as in Equation (4). Then summands of the form $\mathbb{B}(\mathsf{v}_g, \mathsf{x}_{gh^ig^{-1}w'}^0)$ are in one-to-one correspondence with summands of the form $\mathbb{B}(\mathsf{v}_g, \mathsf{x}_{gh^{-i}g^{-1}w''}^0)$.*

Proof. Let w be the word in g, h, g^{-1}, h^{-1} corresponding to the linear part of γ. As is demonstrated in the proof of Lemma 2.4, a summand of the form $\mathbb{B}(\mathsf{v}_g, \mathsf{x}_{gh^ig^{-1}w'}^0)$ may arise from a g factor or a g^{-1} factor.

Case 1: $w = w_1 g w_2$, where g induces a summand of the form $\mathbb{B}(\mathsf{v}_g, \mathsf{x}_{gh^ig^{-1}w'}^0)$. Then $w = w_1 gh^ig^{-1}w_2'$. Thus there is also a summand $\mathbb{B}(w_1 gh^i(-g^{-1}(\mathsf{v}_g), \mathsf{x}_w^0) = \mathbb{B}(\mathsf{v}_g, \mathsf{x}_{gh^{-1}g^{-1}w''}^0)$.

Case 2: $w = w_1 g^{-1} w_2$, where g^{-1} induces a summand of the form $\mathbb{B}(\mathsf{v}_g, \mathsf{x}_{gh^ig^{-1}w'}^0)$. This happens when $w_1 = w_1' gh^{-i}$. Thus there is also a summand $\mathbb{B}(w_1'(\mathsf{v}_g), \mathsf{x}_w^0) = \mathbb{B}(\mathsf{v}_g, \mathsf{x}_{gh^{-i}g^{-1}w_2(w_1')^{-1}}^0)$. $\qquad\square$

Thus every negative summand is potentially cancelled out by a positive summand.

Property C. Let W^+ (resp. W^-) denote the set of all cyclically reduced words of the form $gh^ig^{-1}w$ (resp. $gh^{-i}g^{-1}w$). Let A be the conical interval in A_g^+ bounded by x_{gh}^+ and $g(\mathsf{x}_h^+)$. We will say that the transversal Schottky group $\langle g, h\rangle$ satisfies Property C if for every $\mathsf{x} \in A$:

$$-\mathbb{B}(\mathsf{x}, \mathsf{x}_{w_1}^0) < \mathbb{B}(\mathsf{x}, \mathsf{x}_{w_2}^0), \text{ for all } w_1 \in W^+, w_2 \in W^-.$$

Note that A contains $\mathsf{x}_{gh^i}^+$ for all $i > 0$. Furthermore, we have chosen A so that both terms in the previous inequality are positive.

Property C is more restrictive than is really necessary, but it will make arguments easier. Groups that satisfy Property C are easy find: for instance,

setting $x_g^0 = (1, 0, 0)$ and $\lambda_g = e^{-1}$, and $x_h^0 = (0, 1, 0)$ and $\lambda_h = e^{-2}$, the group $\langle g, h \rangle$ has Property C.

4. Main theorem

Recall the statement of Theorem 1.1.

Theorem 4.1. *Let $G = \langle g, h \rangle$ be a transversal Schottky group having Property C. Then for every $n \geq 1$, G admits an affine deformation $< \gamma, \eta >$ that does not act properly on $\mathbb{A}^{2,1}$, such that the Margulis invariant of every word of length less than or equal to n is positive.*

Proof. Let $n \geq 1$. By choosing appropriate translational parts v_g and v_h, we will exhibit a group $< \gamma, \eta >$, as in the statement of the theorem, such that $\alpha(\gamma \eta^n) < 0$.

First, we choose a translational part v_h for h, such that $\mathbb{B}(v_h, x_w^0) > 0$, for every cyclically reduced word w with terminal letter h.

The set $A \bigcap \mathcal{H}_n \bigcap \mathcal{C}[U^-, U_n^+]$ is non-empty, since it contains the conical interval in A_g^+ bounded by $x_{gh^{n-1}}^+$ and $x_{gh^n}^+$.

Choose any $v \in A \bigcap \mathcal{H}_n \bigcap \mathcal{C}[U^-, U_n^+]$. Lemma 3.6 and Property C ensure that, for any $k > 0$, choosing $v_g = kv$ will yield $\alpha(\gamma) > 0$ for any word of length up to n. But taking k large enough so that $\alpha(\gamma \eta^n) < 0$, it follows that $\langle \gamma, \eta \rangle$ cannot act properly on $\mathbb{A}^{2,1}$. $\square$

References

1. "Ergodic Theory, Symbolic Dynamics and Hyperbolic Spaces, " Bedford, T., Keane, M. and Series, C. eds, Oxford (1991), 369 pages.
2. Charette, V. and Goldman, W., *Affine Schottky groups and crooked tilings,* Proceedings of the Workshop on "Crystallographic Groups and their Generalizations II, "Contemp. Math. (to appear).
3. Drumm, T., *Fundamental polyhedra for Margulis space-times,* Topology **31** (4) (1992), 677-683.
4. ———, *Examples of nonproper affine actions,* Mich. Math. J. **39** (1992), 435–442.
5. Drumm, T. and Goldman, W.,
6. Jones, C., doctoral dissertation, University of Maryland, 2003.
7. Margulis, G., *Free properly discontinuous groups of affine transformations,* Dokl. Akad. Nauk SSSR **272** (1983), 937–940.
8. ———, *Complete affine locally flat manifolds with a free fundamental group,* J. Soviet Math. **134** (1987), 129–134.

Department of Mathematics, University of Manitoba, Winnipeg, Manitoba, Canada
    E-mail address: charette@cc.umanitoba.ca